

"DEPOLARIZATION" OF A POLARIZED PROTON  
BEAM IN A CIRCULAR ACCELERATOR-Appendix

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Appendix Analytical Solution

The solution of the quantum mechanical equation can be expressed in terms of parabolic cylinder functions.

Substituting  $\vec{B}$  from Eq. (6) in Eq. (16) we get

$$\begin{cases} i\dot{\alpha} = -\frac{\Omega}{2} \begin{pmatrix} 1 \\ \alpha + r\beta e^{i\int \omega dt} \end{pmatrix} \\ i\dot{\beta} = -\frac{\Omega}{2} \begin{pmatrix} -1 \\ -\beta + r\alpha e^{-i\int \omega dt} \end{pmatrix} \end{cases} \quad \text{with } \Psi \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix} . \quad (\text{A-1})$$

Let

$$\begin{cases} \alpha = A e^{\frac{i}{2}\int \omega dt} \\ \beta = B e^{-\frac{i}{2}\int \omega dt} \end{cases} \quad (\text{A-2})$$

and we get

$$\begin{cases} \dot{A} = \frac{1}{2}[(\Omega - \omega)A + \Omega r B] \\ \dot{B} = \frac{1}{2}[-(\Omega - \omega)B + \Omega r A] \end{cases} . \quad (\text{A-3})$$

For the simple case, after transformed by Eqs. (10) and (11) this becomes

$$\begin{cases} A' = \frac{1}{2}(\tau A + K B) \\ B' = \frac{1}{2}(K A - \tau B) \end{cases} \quad \text{prime} = \frac{d}{d\tau} . \quad (\text{A-4})$$



Going to the uncoupled second order equations we get

$$\begin{aligned} A'' + \left[ \frac{1}{4}\tau^2 - \left( \frac{1}{2} - \frac{K^2}{4} \right) \right] A &= 0 \\ B'' + \left[ \frac{1}{4}\tau^2 - \left( -\frac{1}{2} - \frac{K^2}{4} \right) \right] B &= 0 \end{aligned} \quad (A-5)$$

These are in the standard forms of parabolic cylinder equations (see e.g. "Handbook of Mathematical Functions", National Bureau of Standards, 9th printing, 1970, p.p. 685-720). Take the A-equation

$$A'' + \left( \frac{1}{4}\tau^2 - \lambda \right) A = 0 \quad \lambda = \frac{1}{2} - \frac{K^2}{4} \quad (A-6)$$

The two standard solutions are the parabolic cylinder functions

$$E(\lambda, \tau) \quad \text{and} \quad E^*(\lambda, \tau) .$$

We are interested in their asymptotic values. For  $\tau^2 \rightarrow \infty$

$$E(\lambda, \tau) \rightarrow \sqrt{\frac{2}{\tau}} e^{i\left(\frac{1}{4}\tau^2 - \lambda \ln \tau + \frac{1}{2}\phi_2 + \frac{1}{4}\pi\right)}$$

where

$$\phi_2 = \arg \Gamma\left(\frac{1}{2} + i\lambda\right) = \arg \Gamma\left(-i\frac{K^2}{4}\right) .$$

With  $\lambda = \frac{1}{2} - \frac{K^2}{4}$  we get at  $\tau \rightarrow \infty$

$$\left\{ \begin{aligned} \lim_{\tau \rightarrow \infty} E\left(\frac{1}{2} - \frac{K^2}{4}, \tau\right) &\rightarrow \sqrt{\frac{2}{\tau}} \sqrt{\tau} e^{\frac{i}{4}(\tau^2 + K^2 \ln \tau + 2\phi_2 + \pi)} \\ &= \sqrt{2} e^{\frac{i}{4}(\tau^2 + K^2 \ln \tau + 2\phi_2 + \pi)} \\ \lim_{\tau \rightarrow -\infty} E^*\left(\frac{1}{2} - \frac{K^2}{4}, \tau\right) &\rightarrow \sqrt{\frac{2}{\tau}} \frac{1}{\sqrt{\tau}} e^{-\frac{i}{4}(\tau^2 + K^2 \ln \tau + 2\phi_2 + \pi)} \rightarrow 0 \end{aligned} \right. \quad (A-7)$$

and at  $\tau \rightarrow -\infty$

$$\begin{cases} \lim_{\tau \rightarrow -\infty} E\left(\frac{i}{2} - \frac{K^2}{4}, \tau\right) \rightarrow \sqrt{2} e^{\frac{\pi}{4}K^2} e^{\frac{i}{4}(\tau^2 + K^2 \ln|\tau| + 2\phi_2 + \pi)} \\ \lim_{\tau \rightarrow -\infty} E^*\left(\frac{i}{2} - \frac{K^2}{4}, \tau\right) \rightarrow 0 \end{cases} \quad (A-8)$$

Our initial condition is

$$|A| = \cos \frac{\theta}{2} = 1 \quad \text{at} \quad \tau = -\infty. \quad (A-9)$$

Hence the solution is

$$A(\tau) = \frac{1}{\sqrt{2}} e^{-\frac{\pi}{4}K^2} E\left(\frac{i}{2} - \frac{K^2}{4}, \tau\right). \quad (A-10)$$

At  $\tau = +\infty$  then

$$|A(+\infty)| \equiv \cos \frac{\Delta\theta}{2} = e^{-\frac{\pi}{4}K^2} \quad (A-11)$$

agreeing with Eq. (13).

It is interesting to note that in the quantum mechanical treatment the wave functions are given by linear equations which frequently have solutions expressible in terms of known functions. In the classical treatment one deals with the amplitudes of the wave functions, hence with more complex nonlinear equations.